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Compositional Semantics
Heinrich Heine University
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Models and Interpretation – Now with Variables![1]

Review: Predicate calculus. Some formulas of Predicate Logic (or 'Predicate Calculus' or 'First order logic'):

(1) $\text{LOVE}(\text{JOHN}, \text{MARY})$ 'John loves Mary'

(2) $\forall x[\text{LOVE}(\text{MARY}, x) \rightarrow \text{HAPPY}(x)]$ 'Everyone whom Mary loves is happy'

(3) $\exists x[\text{EVEN}(x) \wedge x > 1]$ 'There are even numbers greater than 1'

Recall that predicate calculus is a formal language; a logic. Logics have a syntax and a semantics.

Syntax: specifies which expressions of the logic are well-formed, and what their syntactic categories are.

Semantics: specifies which objects the expression correspond to, and what their semantic categories are.

Syntactic categories. Formulas are built up from the following expressions of the syntactic categories including: **individual constants**, **variables**, **predicate constants**, **logical connectives**, and **quantifiers**.

Semantic types. Each expression belongs to a certain semantic type. The types include **individuals** (type e), **truth values** (type t), and **functions** (type $\langle e, t\rangle$).

For our examples we have:

[1] Based closely on lecture notes by V. Borschev and B. Partee

Expressions	Syntactic categories	Semantic Type
JOHN,MARY	(individual) constant	individual
x	variable	individual
HAPPY, EVEN	unary predicate constant	unary relation
LOVE, >	binary predicate constant	binary relation
$\text{LOVE}(\text{JOHN}, \text{MARY})$	(atomic) formula	truth value
$\text{HAPPY}(x)$	(atomic) formula	truth value
$x > 1$	(atomic) formula	truth value
$\forall x[\text{LOVE}(\text{MARY}, x) \rightarrow \text{HAPPY}(x)]$	formula	truth value
$\exists x[\text{EVEN}(x) \wedge x > 1]$	formula	truth value

Interpretation with respect to a model. Expressions of predicate calculus are *interpreted* in *models*. Models consist of a domain of individuals D and an interpretation function I which assigns values to all the constants:

$$\mathbf{M} = \langle D, I\rangle$$

An interpretation function $[\![ ]\!]^M$, built up recursively on the basis of the basic interpreation function I, assigns to every expression α of the language (not just the constants) a **semantic value** $[\![\alpha]\!]^M$.

Here are two models, M_r and M_f (r for "real", and f for "fantasy"/"fiction"/"fake"):

$$M_r = \langle D, I_r\rangle$$

$$M_f = \langle D, I_f\rangle$$

They share the same domain:

$$D = \{m, j, k\}$$

where m, j, and k denote actual individuals whose names are Mary, John and Kim.

In M_r, j is happy, but m and k are not:

$$I_r(\text{HAPPY}) = [\![\text{HAPPY}]\!]^{M_r} = \{j\}$$

In M_f, everybody is happy:

$$I_f(\text{HAPPY}) = [\![\text{HAPPY}]\!]^{M_f} = \{m, j, k\}$$

The interpretation of a constant such as MARY is always m:

$$I_r(\text{MARY}) = [\![\text{MARY}]\!]^{M_r} = m$$

$$I_f(\text{MARY}) = [\![\text{MARY}]\!]^{M_f} = m$$

Some formulae and their truth values:

(4)

		M_f	M_r
a.	HAPPY(JOHN)	true	true
b.	HAPPY(MARY)	true	false
c.	HAPPY(KIM)	true	false
d.	$\forall x$ HAPPY(x)	true	false
e.	$\exists x$ HAPPY(x)	true	true

How can we derive these?

Syntactic categories. Let us use the following:

(5) Terms:

a. individual variables: $x, y, z, x_1, y_1, z_1, x_2, \ldots$

b. individual constants: JOHN, MARY, ...

(6) Predicates:

a. Pred-1: RUN, WALK, HAPPY, CALM, ...EVEN, ODD, ...

b. Pred-2: LOVE, KISS, LIKE, SEE, ...

Composition rules. How to build formulas:

(7) a. If P $\in$ Pred-1 and T $\in$ Term, then P(T) $\in$ Form.

b. If R $\in$ Pred-2 and T_1, T_2 $\in$ Term, then R(T_1,T_2) $\in$ Form.

c. If $\phi \in$ Form , then $\neg\phi \in$ Form.

d. If $\phi \in$ Form and $\psi \in$ Form, then $[\phi \wedge \psi] \in$ Form.

e. If $\phi \in$ Form and $\psi \in$ Form, then $[\phi \vee \psi] \in$ Form.

f. If $\phi \in$ Form and $\psi \in$ Form, then $[\phi \rightarrow \psi] \in$ Form.

g. If $\phi \in$ Form and $\psi \in$ Form, then $[\phi \leftrightarrow \psi] \in$ Form.

h. If v is a variable and $\phi \in$ Form, then $\forall v \phi \in$ Form.

i. If v is a variable and $\phi \in$ Form, then $\exists v \phi \in$ Form.

The last two are new!

Open and closed formulas.

(8) **Free variables**
A **free variable** is a variable that is not bound by any quantifier.

(9) **Open and closed formulas**
A formula is called **open** if it contains free variables.
Otherwise, it is **closed**.

(10)

	Formula	variables	free variables	open or closed?
a.	HAPPY(MARY)	-	-	closed
b.	HAPPY(x)	x	x	open
c.	$\exists x$ HAPPY(x)	x	-	closed
d.	$\exists y$ HAPPY(x)	x, y	x	open
e.	$\exists y$ LOVES(x, y)	x, y	x	open
f.	$\forall x \forall y$ LOVES(x, y)	x, y	-	closed
g.	$[\forall y$ LOVES$(y, x)] \wedge$ HAPPY(z)	x, y, z	x, z	open

Var(ϕ) = the set of all variables of the formula ϕ.

We can define by induction the set FreeVar(ϕ) of *free variables* of the formula ϕ:
FreeVar(ϕ) = Var(ϕ) for every atomic formula ϕ;
FreeVar($\neg\phi$) = FreeVar(ϕ); FreeVar($\phi \wedge \psi$) = FreeVar($\phi \vee \psi$) = FreeVar($\phi \rightarrow \psi$) =
FreeVar($\phi \leftrightarrow \psi$) = FreeVar($\phi$) $\cup$ FreeVar(ψ)
FreeVar($\forall x\phi$) = FreeVar($\exists x\phi$) = FreeVar(ϕ) - $\{v\}$

Semantics. Recall that interpretation is relative to a model.

$$[\![\text{HAPPY}]\!]^{M_r} \neq [\![\text{HAPPY}]\!]^{M_f}$$

A model **M** = $\langle D, I \rangle$ where D is the domain of entities and I is the interpretation function, which assigns semantic values to all constants.

In order to interpret formulas with variables, we need to make interpretation relative to a model *and an assignment function*:

$$[\![\phi]\!]^{M,g}$$

An assignment function assigns individuals to variables. Examples:

(11) $g = \begin{bmatrix} x & \rightarrow & \text{Mary} \\ y & \rightarrow & \text{Mary} \\ z & \rightarrow & \text{Mary} \end{bmatrix}$

(12) $g' = \begin{bmatrix} x & \rightarrow & \text{Mary} \\ y & \rightarrow & \text{John} \\ z & \rightarrow & \text{John} \end{bmatrix}$

(13) Interpretation rules for basic expressions

a. If α is a constant, then $[\![\alpha]\!]^{M,g}$ = I(α).

b. If α is a variable, then $[\![\alpha]\!]^{M,g}$ = g(α).

(14) Interpretation rule for atomic formulae
If $P \in$ Pred-1 and $T \in$ Term, then $[\![P(T)]\!]^{M,g} = 1$ iff $[\![T]\!]^{M,g} \in [\![P]\!]^{M,g}$.

More general rule:
If $R \in$ Pred-n and $T_1, ...T_n \in$ Term, then $[\![R(T_1, ..., T_n)]\!]^{M,g} = 1$ iff

$$\langle [\![T_1]\!]^{M,g}, ..., [\![T_n]\!]^{M,g} \rangle \in [\![R]\!]^{M,g}$$

(15) Interpretation rules for non-atomic formulae

a. $[\![\neg\phi]\!]^{M,g} = 1$ if $[\![\phi]\!]^{M,g} = 0$; otherwise $[\![\neg\phi]\!]^{M,g} = 0$.

b. $[\![\phi \wedge \psi]\!]^{M,g} = 1$ if $[\![\phi]\!]^{M,g} = 1$ and $[\![\psi]\!]^{M,g} = 1$; 0 otherise.

c. Similarly for $[\![\phi \vee \psi]\!]^{M,g}$, $[\![\phi \rightarrow \psi]\!]^{M,g}$, and $[\![\phi \leftrightarrow \psi]\!]^{M,g}$.

(16) Interpretation rules for quantificational formulae

a. $[\![\forall v \phi]\!]^{M,g} = 1$ iff for all $d \in D$, $[\![\phi]\!]^{M,g[d/v]} = 1$

b. $[\![\exists v \phi]\!]^{M,g} = 1$ iff there is a $d \in D$ such that $[\![\phi]\!]^{M,g[d/v]} = 1$

The notation $g[d/x]$ means: The variable assignment which is identitcal to g except for the (possible) difference that $g[d/x]$ assigns the individual d to the variable x.

For example, if:

$g = \begin{bmatrix} x & \rightarrow & \text{Mary} \\ y & \rightarrow & \text{Mary} \\ z & \rightarrow & \text{Mary} \end{bmatrix}$

Then:
$g[\text{John}/y] = \begin{bmatrix} x & \rightarrow & \text{Mary} \\ y & \rightarrow & \text{John} \\ z & \rightarrow & \text{Mary} \end{bmatrix}$

Remember that in M_r Mary is not happy but John is. Thus:

(17) $[\![\text{HAPPY}(y)]\!]^{M_r,g} = 0$

(18) $[\![\text{HAPPY}(y)]\!]^{M_r,g[\text{JOHN}/y]} = 1$

Because of (18), there is a $d \in D$ such that $[\![\text{HAPPY}(y)]\!]^{M_r,g[d/y]} = 1$ (namely John). Hence:

(19) $[\![\exists y\ \text{HAPPY}(y)]\!]^{M_r,g} = 1$